

# AN OPTIMAL GAP THEOREM IN A COMPLETE STRICTLY PSEUDOCONVEX CR $(2n + 1)$ -MANIFOLD

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**ABSTRACT.** In this paper, by applying a linear trace Li-Yau-Hamilton inequality for a positive  $(1, 1)$ -form solution of the CR Hodge-Laplace heat equation and monotonicity of the heat equation deformation, we obtain an optimal gap theorem for a complete strictly pseudocovex CR  $(2n + 1)$ -manifold with nonnegative pseudohermitian bisectional curvature and vanishing torsion. We prove that if the average of the Tanaka-Webster scalar curvature over a ball of radius  $r$  centered at some point  $o$  decays as  $o(r^{-2})$ , then the manifold is flat.

## 1. INTRODUCTION

In [GW1], [S] and [Y], it is conjectured that a complete noncompact Kähler manifold of positive holomorphic bisectional curvature of complex dimension  $m$  is biholomorphic to  $\mathbf{C}^m$ . The first result concerning this conjecture was obtained by Mok-Siu-Yau ([MSY]) and Mok ([Mok2]). Let  $M$  be a complete noncompact Kähler manifold of nonnegative holomorphic bisectional curvature of complex dimension  $m \geq 2$ . They proved that  $M$  is isometrically biholomorphic to  $\mathbf{C}^m$  with the standard flat metric under the assumptions of the maximum volume growth condition

$$V_o(r) \geq \delta r^{2m}$$

for some point  $o \in M$ ,  $\delta > 0$ ,  $r(x) = d(o, x)$  and the scalar curvature  $R$  decays as

$$R(x) \leq \frac{C}{1 + r^{2+\varepsilon}}, \quad x \in M$$

for  $C > 0$  and any arbitrarily small positive constant  $\varepsilon$ . Since then there are several further works aiming to prove the optimal result and reader is referred to [Mok1], [CTZ], [CZ], [N4] and [NT2]. A key common ingredient used in the previous works such as [MSY], [N4] and [NT2] is to solve the so-called Poincare Lelong equation  $\sqrt{-1}\partial\bar{\partial}u = \rho$ , for a given  $d$ -closed real  $(1, 1)$ -form  $\rho$  and then show that  $\text{trace}(\rho) = 0$  by using (1.1). In particular in [NT2], Ni and Tam showed that the solution  $u(x)$  of  $\sqrt{-1}\partial\bar{\partial}u = \text{Ric}$  is of  $o(\log r(x))$  growth with the extra condition  $\liminf_{r \rightarrow \infty} \exp(-ar^2) \int_{B_o(r)} R^2(y) d\mu(y) < \infty$  for some  $a > 0$ . Then the result follows from the Liouville theorem for plurisubharmonic functions which asserts that any continuous plurisubharmonic function with upper growth bound of  $o(\log r(x))$  must be a constant.

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In 2012, L. Ni finally obtained an optimal gap theorem ([N2]) on  $M$  with nonnegative bisectional curvature without the maximum volume growth condition, provided the following scalar decays

$$(1.1) \quad \frac{1}{V_o(r)} \int_{B_o(r)} R(y) d\mu(y) = o(r^{-2}).$$

In the paper of [N2], L. Ni adapted a different method which has also succeeded in the recent resolution of the fundamental gap conjecture in [AC]. The key step is, using a sharp differential estimate and monotonicity of heat equation deformation of positive  $(1, 1)$ -forms as in ([N1]), it provided an alternate argument of proving the above mentioned Liouville theorem.

A Riemannian version of ([MSY]) was proved in [GW2] shortly afterwards. This present paper is concerned with an analogue of CR gap theorem on a complete noncompact strictly pseudoconvex CR  $(2n+1)$ -manifold with nonnegative bisectional curvature. Recently, enlightened by the work of [N1] as above, we obtained the linear trace version of Li-Yau-Hamilton inequality for positive solutions of the CR Lichnerowicz-Laplacian heat equation and then CR monotonicity of heat equation deformation of positive  $(1, 1)$ -forms is available in order to prove the following CR gap Theorem :

**Theorem 1.1.** *Let  $M$  be a complete noncompact strictly pseudoconvex CR  $(2n+1)$ -manifold with nonnegative bisectional curvature and vanishing torsion. Then  $M$  is flat if*

$$(1.2) \quad \frac{1}{V_o(r)} \int_{B_o(r)} S(y) d\mu(y) = o(r^{-2}),$$

for some point  $o \in M$ . Here  $S(y)$  is the Tanaka-Webster scalar curvature and  $V_o(r)$  is the volume of the ball  $B_o(r)$  with respect to the Carnot-Carathéodory distance. As a consequence if  $M$  is not flat, then

$$\liminf_{r \rightarrow \infty} \frac{r^2}{V_o(r)} \int_{B_o(r)} S(y) d\mu(y) > 0$$

for any  $o \in M$ .

Here we adapt the method as in [N2]. Below is the main idea in our proof. We first work on degenerated parabolic systems in CR manifolds which is different to Kähler manifolds :

$$\begin{cases} \frac{\partial}{\partial t} \phi(x, t) &= \Delta_H \phi(x, t), \\ \phi(x, 0) &= Ric(x) \geq 0. \end{cases}$$

Here  $\Delta_H$  is the CR Hodge-Laplacian operator,  $Ric(x) = iR_{\alpha\bar{\beta}}\theta^\alpha \wedge \bar{\theta}^\beta$  is the pseudohermitian Ricci form of a strictly pseudoconvex CR  $(2n+1)$ -manifold.

Let  $M$  be a complete noncompact strictly pseudoconvex CR  $(2n+1)$ -manifold with nonnegative bisectional curvature and vanishing torsion. It follows from Proposition 3.1 that there exists a long time solution  $\phi(x, t)$  with  $\phi(x, t) \geq 0$  on  $M \times [0, \infty)$ . Now let  $u(x, t) = \Lambda(\phi)$  which is nonnegative and satisfies the CR heat equation with  $u(x, 0) = S(x)$ . Li-Yau-Hamilton Harnack quantity (4.6) and monotonicity property (5.13) with vanishing mixed-term implies that  $tu(x, t)$  is nondecreasing in  $t$  for any  $x$ . Finally, the assumption (1.2) and CR moment type estimate (3.10) imply  $\lim_{t \rightarrow \infty} tu(x_0, t) = 0$ . Hence the monotonicity and maximum principle

imply  $tu(x, t) \equiv 0$  for all  $t > 0$  and any  $x \in M$ . The flatness then follows from  $u(x, 0) = 0$  which is clear by continuity.

The rest of the paper is organized as follows. In section 2, we give an introduction to pseudohermitian manifolds and some notations. In section 3, we obtain the CR moment type estimate which is the first key estimate for the proof of main theorem. In section 4, we relate the linear trace Li-Yau-Hamilton type inequality of the CR Lichnerowicz-Laplacian heat equation to a monotonicity formula of the heat solution. In section 5, we prove the CR optimal gap Theorem.

## 2. PRELIMINARY

First we introduce some basic materials in a pseudohermitian  $(2n+1)$ -manifold ( see [L1], [L2] for more details ). Let  $(M, \xi)$  be a  $(2n+1)$ -dimensional, orientable, contact manifold with contact structure  $\xi$ . A CR structure compatible with  $\xi$  is an endomorphism  $J : \xi \rightarrow \xi$  such that  $J^2 = -1$ . We also assume that  $J$  satisfies the following integrability condition: If  $X$  and  $Y$  are in  $\xi$ , then so are  $[JX, Y] + [X, JY]$  and  $J([JX, Y] + [X, JY]) = [JX, JY] - [X, Y]$ .

Let  $\{T, Z_\alpha, Z_{\bar{\alpha}}\}$  be a frame of  $TM \otimes \mathbb{C}$ , where  $Z_\alpha$  is any local frame of  $T_{1,0}$ ,  $Z_{\bar{\alpha}} = \overline{Z_\alpha} \in T_{0,1}$  and  $T$  is the characteristic vector field. Then  $\{\theta, \theta^\alpha, \theta^{\bar{\alpha}}\}$ , which is the coframe dual to  $\{T, Z_\alpha, Z_{\bar{\alpha}}\}$ , satisfies

$$(2.1) \quad d\theta = ih_{\alpha\bar{\beta}}\theta^\alpha \wedge \theta^{\bar{\beta}}$$

for some positive definite hermitian matrix of functions  $(h_{\alpha\bar{\beta}})$ , if we have this contact structure, we call such  $M$  a strictly pseudoconvex CR  $(2n+1)$ -manifold.

The Levi form  $\langle \cdot, \cdot \rangle_{L_\theta}$  is the Hermitian form on  $T_{1,0}$  defined by

$$\langle Z, W \rangle_{L_\theta} = -i \langle d\theta, Z \wedge \overline{W} \rangle.$$

We can extend  $\langle \cdot, \cdot \rangle_{L_\theta}$  to  $T_{0,1}$  by defining  $\langle \overline{Z}, \overline{W} \rangle_{L_\theta} = \overline{\langle Z, W \rangle_{L_\theta}}$  for all  $Z, W \in T_{1,0}$ . The Levi form induces naturally a Hermitian form on the dual bundle of  $T_{1,0}$ , denoted by  $\langle \cdot, \cdot \rangle_{L_\theta^*}$ , and hence on all the induced tensor bundles. Integrating the Hermitian form (when acting on sections) over  $M$  with respect to the volume form  $d\mu = \theta \wedge (d\theta)^n$ , we get an inner product on the space of sections of each tensor bundle.

The pseudohermitian connection of  $(J, \theta)$  is the connection  $\nabla$  on  $TM \otimes \mathbb{C}$  (and extended to tensors) given in terms of a local frame  $Z_\alpha \in T_{1,0}$  by

$$\nabla Z_\alpha = \theta_\alpha^\beta \otimes Z_\beta, \quad \nabla Z_{\bar{\alpha}} = \theta_{\bar{\alpha}}^{\bar{\beta}} \otimes Z_{\bar{\beta}}, \quad \nabla T = 0,$$

where  $\theta_\alpha^\beta$  are the 1-forms uniquely determined by the following equations:

$$\begin{aligned} d\theta^\beta &= \theta^\alpha \wedge \theta_\alpha^\beta + \theta \wedge \tau^\beta, \\ 0 &= \tau_\alpha \wedge \theta^\alpha, \\ 0 &= \theta_\alpha^\beta + \theta_{\bar{\beta}}^{\bar{\alpha}}, \end{aligned}$$

We can write (by Cartan lemma)  $\tau_\alpha = A_{\alpha\gamma}\theta^\gamma$  with  $A_{\alpha\gamma} = A_{\gamma\alpha}$ . The curvature of Webster-Stanton connection, expressed in terms of the coframe  $\{\theta = \theta^0, \theta^\alpha, \theta^{\bar{\alpha}}\}$ , is

$$\begin{aligned}\Pi_\beta^\alpha &= \overline{\Pi_{\bar{\beta}}^{\bar{\alpha}}} = d\omega_\beta^\alpha - \omega_\beta^\gamma \wedge \omega_\gamma^\alpha, \\ \Pi_0^\alpha &= \Pi_\alpha^0 = \Pi_0^{\bar{\beta}} = \Pi_{\bar{\beta}}^0 = \Pi_0^0 = 0.\end{aligned}$$

Webster showed that  $\Pi_\beta^\alpha$  can be written

$$\Pi_\beta^\alpha = R_\beta^\alpha{}_{\rho\bar{\sigma}}\theta^\rho \wedge \theta^{\bar{\sigma}} + W_\beta^\alpha{}_\rho\theta^\rho \wedge \theta - W_{\beta\bar{\rho}}^\alpha\theta^{\bar{\rho}} \wedge \theta + i\theta_\beta \wedge \tau^\alpha - i\tau_\beta \wedge \theta^\alpha$$

where the coefficients satisfy

$$R_{\beta\bar{\alpha}\rho\bar{\sigma}} = \overline{R_{\alpha\bar{\beta}\sigma\bar{\rho}}} = R_{\bar{\alpha}\beta\sigma\rho} = R_{\rho\bar{\alpha}\beta\bar{\sigma}}, \quad W_{\beta\bar{\alpha}\gamma} = W_{\gamma\bar{\alpha}\beta}.$$

Here  $R_\gamma^\delta{}_{\alpha\bar{\beta}}$  is the pseudohermitian curvature tensor,  $R_{\alpha\bar{\beta}} = R_\gamma^\gamma{}_{\alpha\bar{\beta}}$  is the pseudohermitian Ricci curvature tensor,  $S = R_{\alpha\bar{\alpha}}$  is the Tanaka-Webster scalar curvature and  $A_{\alpha\beta}$  is the torsion tensor. Furthermore, we define the bi-sectional curvature

$$R_{\alpha\bar{\alpha}\beta\bar{\beta}}(X, Y) = R_{\alpha\bar{\alpha}\beta\bar{\beta}}X_\alpha X_{\bar{\alpha}}Y_\beta Y_{\bar{\beta}}$$

and the bi-torsion tensor

$$T_{\alpha\bar{\beta}}(X, Y) := i(A_{\bar{\beta}\rho}X_\rho Y_\alpha - A_{\alpha\rho}X_{\bar{\rho}}Y_{\bar{\beta}})$$

and the torsion tensor

$$Tor(X, Y) := h^{\alpha\bar{\beta}}T_{\alpha\bar{\beta}}(X, Y) = i(A_{\bar{\alpha}\rho}X_\rho Y_\alpha - A_{\alpha\rho}X_{\bar{\rho}}Y_{\bar{\alpha}})$$

for any  $X = X_{\bar{\alpha}}Z_\alpha$ ,  $Y = Y_{\bar{\alpha}}Z_\alpha$  in  $T_{1,0}$ .

We will denote components of covariant derivatives with indices preceded by comma; thus write  $A_{\alpha\beta,\gamma}$ . The indices  $\{0, \alpha, \bar{\alpha}\}$  indicate derivatives with respect to  $\{T, Z_\alpha, Z_{\bar{\alpha}}\}$ . For derivatives of a scalar function, we will often omit the comma, for instance,  $u_\alpha = Z_\alpha u$ ,  $u_{\alpha\bar{\beta}} = Z_{\bar{\beta}}Z_\alpha u - \omega_\alpha^\gamma(Z_{\bar{\beta}})Z_\gamma u$ .

For a smooth real-valued function  $u$ , the subgradient  $\nabla_b$  is defined by  $\nabla_b u \in \xi$  and  $\langle Z, \nabla_b u \rangle_{L_\theta} = du(Z)$  for all vector fields  $Z$  tangent to contact plane. Locally  $\nabla_b u = \sum_\alpha u_{\bar{\alpha}}Z_\alpha + u_\alpha Z_{\bar{\alpha}}$ . We also denote  $u_0 = Tu$ .

We can use the connection to define the subhessian as the complex linear map

$$(\nabla^H)^2 u : T_{1,0} \oplus T_{0,1} \rightarrow T_{1,0} \oplus T_{0,1}$$

by

$$(\nabla^H)^2 u(Z) = \nabla_Z \nabla_b u.$$

In particular,

$$|\nabla_b u|^2 = 2u_\alpha u_{\bar{\alpha}}, \quad |\nabla_b^2 u|^2 = 2(u_{\alpha\beta} u_{\bar{\alpha}\bar{\beta}} + u_{\alpha\bar{\beta}} u_{\bar{\alpha}\beta}).$$

Also

$$\Delta_b u = Tr((\nabla^H)^2 u) = \sum_\alpha (u_{\alpha\bar{\alpha}} + u_{\bar{\alpha}\alpha}).$$

The Kohn-Rossi Laplacian  $\square_b$  on functions is defined by

$$\square_b \varphi = 2\bar{\partial}_b^* \bar{\partial}_b \varphi = (\Delta_b + inT)\varphi = -2\varphi_{\bar{\alpha}}^{\bar{\alpha}}$$

and on  $(p, q)$ -forms is defined by

$$\square_b = 2(\bar{\partial}_b^* \bar{\partial}_b + \bar{\partial}_b \bar{\partial}_b^*).$$

Next we recall the following commutation relations ([L1]). Let  $\varphi$  be a scalar function and  $\sigma = \sigma_\alpha \theta^\alpha$  be a  $(1, 0)$  form, then we have

$$\begin{aligned} \varphi_{\alpha\beta} &= \varphi_{\beta\alpha}, \\ \varphi_{\alpha\bar{\beta}} - \varphi_{\bar{\beta}\alpha} &= ih_{\alpha\bar{\beta}} \varphi_0, \\ \varphi_{0\alpha} - \varphi_{\alpha 0} &= A_{\alpha\beta} \varphi_{\bar{\beta}}, \\ \sigma_{\alpha,0\beta} - \sigma_{\alpha,\beta 0} &= \sigma_{\alpha,\bar{\gamma}} A_{\gamma\beta} - \sigma_{\gamma} A_{\alpha\beta,\bar{\gamma}}, \\ \sigma_{\alpha,0\bar{\beta}} - \sigma_{\alpha,\bar{\beta} 0} &= \sigma_{\alpha,\gamma} A_{\bar{\gamma}\bar{\beta}} + \sigma_{\gamma} A_{\bar{\gamma}\bar{\beta},\alpha}, \end{aligned}$$

and

$$\begin{aligned} \sigma_{\alpha,\beta\gamma} - \sigma_{\alpha,\gamma\beta} &= iA_{\alpha\gamma} \sigma_\beta - iA_{\alpha\beta} \sigma_\gamma, \\ \sigma_{\alpha,\bar{\beta}\bar{\gamma}} - \sigma_{\alpha,\bar{\gamma}\bar{\beta}} &= ih_{\alpha\bar{\beta}} A_{\bar{\gamma}\bar{\rho}} \sigma_\rho - ih_{\alpha\bar{\gamma}} A_{\bar{\beta}\bar{\rho}} \sigma_\rho, \\ \sigma_{\alpha,\beta\bar{\gamma}} - \sigma_{\alpha,\bar{\gamma}\beta} &= ih_{\beta\bar{\gamma}} \sigma_{\alpha,0} + R_{\alpha\bar{\rho}\beta\bar{\gamma}} \sigma_\rho. \end{aligned}$$

Moreover for multi-index  $I = (\alpha_1, \dots, \alpha_p)$ ,  $\bar{J} = (\bar{\beta}_1, \dots, \bar{\beta}_q)$ , we denote  $I(\alpha_k = \mu) = (\alpha_1, \dots, \alpha_{k-1}, \mu, \alpha_{k+1}, \dots, \alpha_p)$ . Then

$$\begin{aligned} \eta_{I\bar{J},\mu\lambda} - \eta_{I\bar{J},\lambda\mu} &= i \sum_{k=1}^p (\eta_{I(\alpha_k=\mu)\bar{J}} A_{\alpha_k\lambda} - \eta_{I(\alpha_k=\lambda)\bar{J}} A_{\alpha_k\mu}) \\ &\quad - i \sum_{k=1}^q (\eta_{I\bar{J}(\bar{\beta}_k=\bar{\gamma})} h_{\bar{\beta}_k\mu} A_{\lambda}^{\bar{\gamma}} - \eta_{I\bar{J}(\bar{\beta}_k=\bar{\gamma})} h_{\bar{\beta}_k\lambda} A_{\mu}^{\bar{\gamma}}), \end{aligned}$$

and

$$\begin{aligned} \eta_{I\bar{J},\lambda\bar{\mu}} - \eta_{I\bar{J},\bar{\mu}\lambda} &= ih_{\lambda\bar{\mu}} \eta_{I\bar{J},0} + \sum_{k=1}^p \eta_{I(\alpha_k=\gamma)\bar{J}} R_{\alpha_k}{}^{\gamma}{}_{\lambda\bar{\mu}} + \sum_{k=1}^q \eta_{I\bar{J}(\bar{\beta}_k=\bar{\gamma})} R_{\bar{\beta}_k}{}^{\bar{\gamma}}{}_{\lambda\bar{\mu}} \\ \eta_{I\bar{J},0\mu} - \eta_{I\bar{J},\mu 0} &= A_{\mu}^{\bar{\rho}} \eta_{I\bar{J},\bar{\rho}} - \sum_{k=1}^p A_{\alpha_k\mu,\bar{\rho}} \eta_{I(\alpha_k=\rho)\bar{J}} + \sum_{k=1}^q A_{\mu\rho,\bar{\beta}_k} \eta_{I\bar{J}(\bar{\beta}_k=\bar{\rho})}. \end{aligned}$$

Finally, we recall the following definition.

**Definition 2.1.** A piecewise smooth curve  $\gamma : [0, 1] \rightarrow M$  is said to be horizontal if  $\gamma'(t) \in \xi$  whenever  $\gamma'(t)$  exists. The length of  $\gamma$  is then defined by

$$l(\gamma) = \int_0^1 \langle \gamma'(t), \gamma'(t) \rangle_{L_\theta}^{\frac{1}{2}} dt.$$

The Carnot-Carathéodory distance between two points  $p, q \in M$  is

$$d_c(p, q) = \inf \{ l(\gamma) \mid \gamma \in C_{p,q} \},$$

where  $C_{p,q}$  is the set of all horizontal curves joining  $p$  and  $q$ .

### 3. CR MOMENT-TYPE ESTIMATES

Let  $(M, J, \theta)$  be a strictly pseudoconvex CR  $(2n+1)$ -manifold. In our recent paper ([CCT] and [CCF]), we consider the CR Hodge-Laplacian

$$\Delta_H = -\frac{1}{2}(\square_b + \bar{\square}_b)$$

for Kohn-Rossi Laplacian  $\square_b$ . For any  $(1, 1)$ -form  $\phi(x, t) = \phi_{\alpha\bar{\beta}}\theta^\alpha \wedge \bar{\theta}^\beta$ , we study the CR Hodge-Laplacian heat equation on  $M \times [0, T)$

$$(3.1) \quad \frac{\partial}{\partial t}\phi(x, t) = \Delta_H\phi(x, t).$$

It follows from the CR Bochner-Weitzenbock Formula ([CCF]) that the CR parabolic equation (3.1) is equivalent to the CR analogue of Lichnerowicz-Laplacian heat equation :

$$(3.2) \quad \frac{\partial}{\partial t}\phi_{\alpha\bar{\beta}} = \Delta_b\phi_{\alpha\bar{\beta}} + 2R_{\alpha\bar{\gamma}\mu\bar{\beta}}\phi_{\gamma\bar{\mu}} - (R_{\gamma\bar{\beta}}\phi_{\alpha\bar{\gamma}} + R_{\alpha\bar{\gamma}}\phi_{\gamma\bar{\beta}}).$$

In this section, we consider the following Dirichlet problem of degenerate parabolic systems :

$$(3.3) \quad \begin{cases} \left(\frac{\partial}{\partial t} - \Delta_H\right)\phi &= 0, & \text{on } \Omega \times [0, \infty), \\ \phi(x, t) &= 0, & \text{on } \partial\Omega \times [0, \infty), \\ \phi(x, 0) &= \phi_{ini}(x) & \text{on } \Omega. \end{cases}$$

In contrast to Kähler case, the regularity of a solution for  $\Delta_H$  up to  $\partial\Omega$  may depend on geometry around the characteristic point at the boundary ([J1] and [J2]) in the CR setting. In fact,

**Proposition 3.1.** *There exists "sweetsop" exhaustion domains  $\Omega_\mu$  such that the solutions  $\phi_\mu$  of (3.3) are  $C(C^{2,\alpha}(\bar{\Omega}_\mu, \Lambda^{1,1}), [0, T))$ .*

We will give a detail proof of Proposition 3.1 in Appendix A. After the construction of the "sweetsop" exhaustion domain  $\Omega_\mu$  for  $\Delta_H$  as in Proposition 3.1, one is able to apply semigroup method ([P]) to obtain better regularity of the solution of the CR Lichnerowicz-Laplacian heat equation (3.3) which depends on regularity of the initial condition. One more tensor maximum principle below is needed in the proof of main theorem in order to have nonnegativity of the constructed solution  $\phi_\mu$  if the initial data is nonnegative.

**Proposition 3.2.** *Let  $(M, J, \theta)$  be a strictly pseudoconvex CR  $(2n+1)$ -manifold with non-negative bisectonal curvature. Let  $\Omega$  be bounded domain in  $M$ . Assume that  $\phi(x, t)$  is a  $(1, 1)$ -form satisfies*

$$\begin{cases} \left(\frac{\partial}{\partial t} - \Delta_H\right)\phi &= 0, & \text{on } \Omega \times [0, \infty), \\ \phi(x, t) &\geq 0, & \text{on } \partial\Omega \times [0, \infty), \\ \phi(x, 0) &\geq 0 & \text{on } \Omega. \end{cases}$$

*Then  $\phi(x, t) \geq 0$  on  $\Omega \times [0, \infty)$ .*

**Proof.** Similar to proposition 11.1 in [NN]. □

The first key estimate for the proof of main theorem is the moment type estimate. This estimate is first introduced by L. Ni ([N3]). By using Li-Yau type heat kernel estimate, he proved that a nonnegative solution  $u(x, t)$  of the heat equation are  $t^{d/2}$  growth if and only if the average function  $k(x, r) := \frac{1}{V(r)} \int_{B_x(r)} f(y) dy$  of the initial data  $f(y)$  grows as  $r^d$  in a certain complete Kaehler manifold. In our CR setting, we only has the CR moment type estimate for a nonnegative heat solution which can be express as  $P_t f$  for a smooth bounded

function  $f$  on  $M$ . In contrast to Kähler case, in general, we do not know if any nonnegative heat solution could hold.

To introduce our version, we will follow from semigroup method as in [M] (also [BBGM]). It is known that the heat semigroup  $(P_t)_{t \geq 0}$  is given by

$$P_t = \int_0^\infty e^{-\lambda t} dE_\lambda$$

for the spectral decomposition of  $\Delta_b = -\int_0^\infty \lambda dE_\lambda$  in  $L^2(M)$ . It is a one-parameter family of bounded operators on  $L^2(M)$ . We denote

$$P_t f(x) = \int_M p(x, y, t) f(y) d\mu(y),$$

for  $f \in C_0^\infty(M)$ . Here  $p(x, y, t) > 0$  is the so-called symmetric heat kernel associated to  $P_t$ . Due to hypoellipticity of  $\Delta_b$ , the function  $(x, t) \rightarrow P_t f(x)$  is smooth on  $M \times (0, \infty)$ .

In the following we use  $V(r)$  and  $B_x(r)$  denote the volume of a unit ball with respect to the Carnot-Carathéodory distance and measure  $d\mu = \theta \wedge (d\theta)^n$ . We recall some facts from [M] (also [BG] and [BBGM]). For  $f, g, h \in C^\infty(M)$ , we define

(i)

$$\Gamma(f, g) = \frac{1}{2} \Delta_b(fg) - f \Delta_b g - g \Delta_b f.$$

(ii)

$$\Gamma_2(f, g) = \frac{1}{2} [\Delta_b \Gamma(f, g) - \Gamma(f, \Delta_b g) - \Gamma(g, \Delta_b f)].$$

(iii)

$$\Gamma^Z(fg, h) = f \Gamma^Z(g, h) + g \Gamma^Z(f, h).$$

(iv)

$$\Gamma_2^Z(f, g) = \frac{1}{2} [\Delta_b \Gamma^Z(f, g) - \Gamma^Z(f, \Delta_b g) - \Gamma^Z(g, \Delta_b f)].$$

Here we denote  $\Gamma(f) = \Gamma(f, f)$ ,  $\Gamma_2(f) = \Gamma_2(f, f)$ ,  $\Gamma^Z(f) = \Gamma^Z(f, f)$  and  $\Gamma_2^Z(f) = \Gamma_2^Z(f, f)$ . Note that in a complete strictly pseudoconvex CR  $(2n+1)$ -manifold with vanishing torsion. One can have  $\Gamma(f, f) = (\nabla_b f, \nabla_b f)$  and  $\Gamma_2(f) = \|\nabla_b^2 f\|^2 + Ric(\nabla_b f, \nabla_b f) + \frac{n}{2} \|\nabla_T \nabla_b f\|^2$  and  $\Gamma^Z(f, g) = (\nabla_T f, \nabla_T g)$ .

**Definition 3.1.** We say that  $(M, J, \theta)$  satisfies the generalized curvature-dimension inequality  $CD(\rho_1, \rho_2, \kappa, d)$  with respect to  $\Delta_b$  if there exist constants  $\rho_1$  a real number,  $\rho_2 > 0$ ,  $\kappa \geq 0$ , and  $d \geq 2$  such that the inequality

$$\Gamma_2(f) + \nu \Gamma_2^Z(f) \geq \frac{1}{d} (\Delta_b f)^2 + \left(\rho_1 - \frac{\kappa}{\nu}\right) \Gamma(f) + \rho_2 \Gamma^Z(f)$$

holds for every  $f \in C^\infty(M)$  and every  $\nu > 0$ .

We define

$$(3.4) \quad D := d \left( 1 + \frac{3\kappa}{2\rho_2} \right)$$

and

$$\rho_1^- = \max(-\rho_1, 0).$$

**Lemma 3.1.** (i) ([M, Theorem 4]) *Let  $(M, J, \theta)$  be a complete strictly pseudoconvex CR  $(2n+1)$ -manifold of vanishing torsion with*

$$\text{Ric} \geq \rho_1.$$

*Then  $M$  satisfies the generalized curvature-dimension inequality  $CD(\rho_1, \frac{n}{2}, 1, 2n)$  with  $\rho_2 = \frac{n}{2}, \kappa = 1$  and  $d = 2n$ . Moreover for any given  $R_0 > 0$ , there exists a constant  $C(d, \kappa, \rho_2) > 0$  such that*

$$\mu(B(x, R)) \leq C(d, \kappa, \rho_2) \frac{\exp(2d\rho_1^- R_0^2)}{R_0^D p(x, x, R_0^2)} R^D \exp(2d\rho_1^- R^2)$$

*for every  $x \in M$  and  $R \geq R_0$ . In particular if  $M$  is a complete strictly pseudoconvex CR  $(2n+1)$ -manifold of nonnegative Ricci curvature and vanishing torsion, then there exists a constant  $C_1 > 0$  such that*

$$(3.5) \quad \mu(B(x, R)) \leq \frac{C_1}{R_0^D p(x, x, R_0^2)} R^D$$

*for  $R \geq R_0$ .*

(ii) ([BG]) *Let  $(M, J, \theta)$  be a complete strictly pseudoconvex CR  $(2n+1)$ -manifold of nonnegative Ricci curvature and vanishing torsion. Then, for any  $\varepsilon > 0$ , there exists a constant  $C_3(d, \rho_2, \kappa, \varepsilon) > 0$  such that*

$$(3.6) \quad p(x, y, t) \leq \frac{C(d, \rho_2, \kappa, \varepsilon)}{\mu(B(x, \sqrt{t}))^{\frac{1}{2}} \mu(B(y, \sqrt{t}))^{\frac{1}{2}}} \exp\left(-\frac{d^2(x, y)}{(4 + \varepsilon)t}\right).$$

(iii) ([BBGM]) *Let  $(M, J, \theta)$  be a complete strictly pseudoconvex CR  $(2n+1)$ -manifold of nonnegative Ricci curvature and vanishing torsion. Then there exists a constant  $C_2 > 0$  such that*

$$(3.7) \quad p(x, x, 2R^2) \geq \frac{C_2}{\mu(B(x, R))}.$$

**Remark 3.1.** *Let  $(M, J, \theta)$  be a complete strictly pseudoconvex CR  $(2n+1)$ -manifold of nonnegative Ricci curvature and vanishing torsion. (3.5) and (3.7) together imply the doubling property. That is*

$$(3.8) \quad \mu(B(x, R)) \leq \frac{C_1}{R_0^D p(x, x, R_0^2)} R^D \leq C\left(\frac{R}{R_0}\right)^D \mu\left(B\left(x, \frac{R_0}{\sqrt{2}}\right)\right).$$

*By taking  $R_0 = \frac{R}{\sqrt{2}}$ , then there exists a constant  $C_4 > 0$  such that*

$$(3.9) \quad \mu(B(x, R)) \leq C_4(n, D) \mu\left(B\left(x, \frac{R}{2}\right)\right).$$

Applying above Lemma 3.1, we are able to prove the following moment type estimate for those solution of form  $P_t f$ .

**Theorem 3.1.** *Let  $(M, J, \theta)$  be a complete strictly pseudoconvex CR  $(2n+1)$ -manifold of nonnegative Ricci curvature and vanishing torsion. Assume that  $u$  is a solution of CR heat equation*

$$\frac{\partial}{\partial t} u = \Delta_b u$$



such that

$$u(x, t) = P_t f$$

for a nonnegative bounded function  $f$ . Assume that for any  $a > -D - 2$  (where  $D = 2n + 6$  is defined in 3.4), we have

$$\frac{1}{V(r)} \int_{B_x(r)} f(y) d\mu(y) \leq Ar^a$$

for a constant  $A > 0$  and  $r \geq R \geq 1$ . Then there exists a constant  $C(n, d)$  such that

$$(3.10) \quad u(x, t) \leq C(n, d) At^{\frac{a}{2}}$$

for all  $t \geq R^2$ .

**Proof.** Let  $\delta = \frac{d(x, y)}{\sqrt{t}}$ . Thus

$$(3.11) \quad B_x(\sqrt{t}) \subseteq B_y((\delta + 1)\sqrt{t}).$$

It follow from (3.9) and (3.11) that

$$V_x(\sqrt{t}) \leq V_y((\delta + 1)\sqrt{t}) \leq C(d, \kappa, \rho_2)(\delta + 1)^D V_y(\sqrt{t}).$$

That is,

$$(3.12) \quad \frac{V_x(\sqrt{t})}{V_y(\sqrt{t})} \leq C(d, \kappa, \rho_2)(\delta + 1)^D.$$

We can rewrite (3.6) as

$$(3.13) \quad \begin{aligned} p(x, y, t) &\leq \frac{C(d, \rho_2, \kappa, \varepsilon)}{\mu(B(x, \sqrt{t}))^{\frac{1}{2}} \mu(B(y, \sqrt{t}))^{\frac{1}{2}}} \exp\left(-\frac{d^2(x, y)}{(4+\varepsilon)t}\right) \\ &\leq \frac{C(d, \rho_2, \kappa, \varepsilon)}{\mu(B(x, \sqrt{t}))} \left(\frac{\mu(B(x, \sqrt{t}))}{\mu(B(y, \sqrt{t}))}\right)^{\frac{1}{2}} \exp\left(-\frac{d^2(x, y)}{(4+\varepsilon)t}\right) \\ &\leq \frac{C(d, \rho_2, \kappa, \varepsilon)}{\mu(B(x, \sqrt{t}))} \exp\left(-\frac{d^2(x, y)}{(4+\varepsilon)t}\right). \end{aligned}$$

Then, based on (3.12) and (3.13), Theorem 3.1 follows from the proof of Theorem 3.1 in [N3] in case of  $u(x, t) = P_t f$  for a nonnegative bounded function  $f$ . The use of volume comparison can be replace by (3.8).  $\square$

#### 4. CR LINEAR TRACE LI-YAU-HAMILTON TYPE INEQUALITY

In this section, we first relate the linear trace Li-Yau-Hamilton type inequality of the CR Lichnerowicz-Laplacian heat equation to a monotonicity formula of the heat solution. More precisely, let  $\eta_{\alpha\bar{\beta}}(x, t)$  be a symmetric  $(1, 1)$  tensor satisfying the CR Lichnerowicz-Laplacian heat equation

$$(4.1) \quad \frac{\partial}{\partial t} \eta_{\alpha\bar{\beta}} = \Delta_b \eta_{\alpha\bar{\beta}} + 2R_{\alpha\bar{\gamma}\mu\bar{\beta}} \eta_{\gamma\bar{\mu}} - (R_{\gamma\bar{\beta}} \eta_{\alpha\bar{\gamma}} + R_{\alpha\bar{\gamma}} \eta_{\gamma\bar{\beta}})$$

on  $M \times [0, T)$ . As in the paper of [CCF], we define following Harnack quantity

$$Z(x, t)(V) := k_1 \left( \frac{1}{2} \left( (\operatorname{div} \eta)_{\alpha, \bar{\alpha}} + (\operatorname{div} \eta)_{\bar{\alpha}, \alpha} \right) + (\operatorname{div} \eta)_{\alpha} V_{\bar{\alpha}} + (\operatorname{div} \eta)_{\bar{\alpha}} V_{\alpha} + V_{\bar{\alpha}} V_{\beta} \eta_{\alpha\bar{\beta}} \right) + \frac{H}{t}$$

for any vector field  $V \in T^{1,0}(M)$ ,  $H = h^{\alpha\bar{\beta}}\eta_{\alpha\bar{\beta}}$  and  $0 < k_1 \leq 8$ . We proved

**Theorem 4.1.** ([CCF]) *Let  $(M, J, \theta)$  be a complete strictly pseudoconvex CR  $(2n+1)$ -manifold of nonnegative bisectional curvature and vanishing torsion. Let  $\eta_{\alpha\bar{\beta}}(x, t)$  be a symmetric  $(1, 1)$  tensor satisfying the CR Lichnerowicz-Laplacian heat equation (4.1) on  $M \times (0, T)$  with*

$$\eta_{\alpha\bar{\beta}}(x, 0) \geq 0,$$

and

$$\nabla_T \eta(x, 0) = 0.$$

Then

$$Z(x, t) \geq 0$$

on  $M \times (0, T)$  for any  $(1, 0)$  vector field  $V$  and  $0 < k_1 \leq 8$  if there exists constant  $a > 0$  such that

$$(4.2) \quad \int_0^T \int_M e^{-ar^2} \|\eta(x, t)\|^2 d\mu dt < \infty,$$

$$(4.3) \quad \int_0^T \int_M e^{-ar^2} \|\nabla_T \eta(x, t)\|^2 d\mu dt < \infty,$$

$$(4.4) \quad \int_M e^{-ar^2} \|\eta(x, 0)\| d\mu < \infty.$$

Let  $\phi$  be a  $(p, q)$ -form. Define contraction operator  $\Lambda : \Lambda^{p,q} \rightarrow \Lambda^{p-1,q-1}$  as follow

$$(\Lambda\phi)_{\alpha_1 \dots \alpha_{p-1} \bar{\beta}_1 \dots \bar{\beta}_{q-1}} = \frac{1}{\sqrt{-1}} (-1)^{p-1} h^{\alpha\bar{\beta}} \phi_{\alpha\alpha_1 \dots \alpha_{p-1} \bar{\beta}\bar{\beta}_1 \dots \bar{\beta}_{q-1}}.$$

Then it is a straightforward computation, we have

**Lemma 4.1.** ([CCT]) *Let  $(M, J, \theta)$  be a strictly pseudoconvex CR  $(2n+1)$ -manifold. We have the Kähler type identities*

(i)

$$[\partial_b, \Lambda] = -\sqrt{-1}\bar{\partial}_b^* \quad \text{and} \quad [\bar{\partial}_b, \Lambda] = \sqrt{-1}\partial_b^*.$$

(ii)

$$[\bar{\partial}_b, \square_b] = 2iT\bar{\partial}_b \quad \text{and} \quad [\partial_b, \square_b] = 0.$$

(iii)

$$[\bar{\partial}_b, \Delta_H] = -iT\bar{\partial}_b \quad \text{and} \quad [\Lambda, \Delta_H] = 0.$$

**Lemma 4.2.** *Let  $\phi$  be a nonnegative  $(1, 1)$ -form. Define  $Q(\phi, V)$  as*

$$(4.5) \quad Q(\phi, V, k_2) = k_2 \left( \frac{1}{2\sqrt{-1}} (\bar{\partial}_b^* \partial_b^* - \partial_b^* \bar{\partial}_b^*) \phi + \frac{1}{\sqrt{-1}} (\bar{\partial}_b^* \phi)_V - \frac{1}{\sqrt{-1}} (\partial_b^* \phi)_V + \phi_{V, \bar{V}} \right) + \frac{\Lambda\phi}{t}.$$

Then this is equivalent to

$$Q(\eta, V, k_2) = k_2 \left( \frac{1}{2} \left( (\operatorname{div} \eta)_{\alpha, \bar{\alpha}} + (\operatorname{div} \eta)_{\bar{\alpha}, \alpha} \right) + (\operatorname{div} \eta)_\alpha V_{\bar{\alpha}} + (\operatorname{div} \eta)_{\bar{\alpha}} V_\alpha + \eta_{\alpha\bar{\beta}} V_\alpha V_{\bar{\beta}} \right) + \frac{H}{t}$$

for a symmetric  $(1, 1)$ -tensor  $\eta_{\alpha\bar{\beta}} := \frac{1}{\sqrt{-1}} \phi_{\alpha\bar{\beta}}$ . In particular, by taking  $V = 0$ , and  $k_2 = 2$ , we have

$$(4.6) \quad Q(\phi, V) = -\Delta_H \Lambda\phi + (\bar{\partial}_b^* \Lambda \bar{\partial}_b + \operatorname{conj}) \phi + \frac{u}{t}$$

for  $u = \Lambda\phi$ .

**Proof.** As in [CCF], we have the formula for a  $(p, q+1)$ -form  $\psi$

$$(\bar{\partial}_b^* \psi)_{\alpha_1 \dots \alpha_p \bar{\beta}_1 \dots \bar{\beta}_q} = (-1)^p \frac{1}{q+1} \sum_{i=1}^{q+1} (-1)^i \nabla_{\mu} \psi_{\alpha_1 \dots \alpha_p \bar{\beta}_1 \dots \bar{\beta}_{i-1} \bar{\mu} \bar{\beta}_i \dots \bar{\beta}_q}$$

and a  $(p+1, q)$ -form  $\varphi$

$$(\partial_b^* \varphi)_{\alpha_1 \dots \alpha_p \bar{\beta}_1 \dots \bar{\beta}_q} = (-1) \frac{1}{p+1} \nabla_{\bar{\mu}} \varphi_{\mu \alpha_1 \dots \alpha_p \bar{\beta}_1 \dots \bar{\beta}_q}.$$

Thus for a  $(1, 1)$ -form  $\phi$ , we have

$$(\partial_b^* \phi)_{\bar{\gamma}} = -\nabla_{\bar{\mu}} \phi_{\mu \bar{\gamma}}$$

and

$$\bar{\partial}_b^* \partial_b^* \phi = \nabla_{\gamma} (\nabla_{\bar{\mu}} \phi_{\mu \bar{\gamma}}).$$

Then the first term of (4.5) become

$$\begin{aligned} \frac{1}{2\sqrt{-1}} (\bar{\partial}_b^* \partial_b^* - \partial_b^* \bar{\partial}_b^*) \phi &= \frac{1}{2\sqrt{-1}} \bar{\partial}_b^* \partial_b^* \phi + conj. \\ &= \frac{1}{2\sqrt{-1}} \nabla_{\gamma} (\nabla_{\bar{\mu}} \phi_{\mu \bar{\gamma}}) + conj. \\ &= \frac{1}{2} ((\operatorname{div} \eta)_{\alpha, \bar{\alpha}} + conj.) \end{aligned}$$

We are done. On the other hand, taking  $V = 0$  and  $k_2 = 2$ , by lemma 4.1 we have

$$\begin{aligned} \frac{2}{2\sqrt{-1}} (\bar{\partial}_b^* \partial_b^* - \partial_b^* \bar{\partial}_b^*) \phi &= \bar{\partial}_b^* [\bar{\partial}_b, \Lambda] \phi - \partial_b^* [\partial_b, \Lambda] \phi \\ &= -\bar{\partial}_b^* \bar{\partial}_b \Lambda \phi - \partial_b^* \partial_b \Lambda \phi + \bar{\partial}_b^* \Lambda \bar{\partial}_b \phi + \partial_b^* \Lambda \partial_b \phi \\ &= -\Delta_H \Lambda \phi + \bar{\partial}_b^* \Lambda \bar{\partial}_b \phi + \partial_b^* \Lambda \partial_b \phi. \end{aligned}$$

Here we use the fact that  $\bar{\partial}_b^* f = \partial_b^* f = 0$  for any scalar function. Then formula (4.6) follows.  $\square$

**Remark 4.1.** The regularity of the heat solution in Proposition 3.1 and the following Lemma is used to prove the "mix-term"  $(\bar{\partial}_b^* \Lambda \bar{\partial}_b + conj) \phi$  in (4.6) vanishing as in (5.13) and (5.14) which is the key step in the proof of our main theorem.

**Lemma 4.3.** Let  $(M, J, \theta)$  be a complete strictly pseudoconvex CR  $(2n+1)$ -manifold with nonnegative bisectional curvature and vanishing torsion. Let  $\phi$  be a solution of the CR Hodge-Laplace heat equation (3.2). Then  $\|\Lambda \bar{\partial}_b \phi\|$  satisfies

$$\left( \frac{\partial}{\partial t} - \Delta_b \right) \|\Lambda \bar{\partial}_b \phi\| \leq \|\Lambda T \bar{\partial}_b \phi\|.$$

**Proof.** We have the formula for a  $(p, q)$ -form  $\psi$

$$(\bar{\partial}_b \psi)_{\alpha_1 \dots \alpha_p \bar{\beta}_1 \dots \bar{\beta}_{q+1}} = (-1)^p \sum_{i=1}^{q+1} (-1)^{i-1} \nabla_{\bar{\beta}_i} \psi_{\alpha_1 \dots \alpha_p \bar{\beta}_1 \dots \bar{\beta}_{i-1} \bar{\beta}_{i+1} \dots \bar{\beta}_{q+1}}.$$

So that

$$(\bar{\partial}_b \phi)_{\alpha \bar{\beta} \bar{\gamma}} = -\nabla_{\bar{\beta}} \phi_{\alpha \bar{\gamma}} + \nabla_{\bar{\gamma}} \phi_{\alpha \bar{\beta}}$$

and

$$\begin{aligned} (\Lambda \bar{\partial}_b \phi)_{\bar{\gamma}} &= ih^{\alpha \bar{\beta}} \nabla_{\bar{\beta}} \phi_{\alpha \bar{\gamma}} - ih^{\alpha \bar{\beta}} \nabla_{\bar{\gamma}} \phi_{\alpha \bar{\beta}} \\ &= h^{\alpha \bar{\beta}} \nabla_{\bar{\beta}} \eta_{\alpha \bar{\gamma}} - h^{\alpha \bar{\beta}} \nabla_{\bar{\gamma}} \eta_{\alpha \bar{\beta}} \\ &= (\operatorname{div} \eta)_{\bar{\gamma}} - \nabla_{\bar{\gamma}} u. \end{aligned}$$

Note that  $\Lambda \bar{\partial}_b \phi$  satisfies the CR Hodge Laplace heat equation, i.e.,

$$\begin{aligned} \left(\frac{\partial}{\partial t} + \Delta_H\right) \Lambda \bar{\partial}_b \phi &= -\Lambda \bar{\partial}_b \Delta_H \phi + \Delta_H \Lambda \bar{\partial}_b \phi \\ &= -\Lambda \Delta_H \bar{\partial}_b \phi + \Delta_H \Lambda \bar{\partial}_b \phi + i\Lambda T \bar{\partial}_b \phi \\ &= [\Delta_H, \Lambda] \bar{\partial}_b \phi + i\Lambda T \bar{\partial}_b \phi \\ &= i\Lambda T \bar{\partial}_b \phi. \end{aligned}$$

Hence we have

$$\begin{aligned} \left(\frac{\partial}{\partial t} - \Delta_b\right) \sqrt{\|\Lambda \bar{\partial}_b \phi\|^2} &= \frac{\Lambda \bar{\partial}_b \phi}{\|\Lambda \bar{\partial}_b \phi\|} \cdot (-\Delta_H \Lambda \bar{\partial}_b \phi - \Delta_b \Lambda \bar{\partial}_b \phi) + \frac{\Lambda \bar{\partial}_b \phi}{\|\Lambda \bar{\partial}_b \phi\|} \cdot i\Lambda T \bar{\partial}_b \phi \\ &= -\frac{1}{\|\Lambda \bar{\partial}_b \phi\|} R_{\alpha\bar{\beta}} (\Lambda \bar{\partial}_b \phi)_\alpha \overline{(\Lambda \bar{\partial}_b \phi)_{\bar{\beta}}} + \frac{\Lambda \bar{\partial}_b \phi}{\|\Lambda \bar{\partial}_b \phi\|} \cdot i\Lambda T \bar{\partial}_b \phi \end{aligned}$$

where in second line we use formula (3.1) of [CCF] for  $(1, 0)$ -form  $\Lambda \bar{\partial}_b \phi$ .  $\square$

Before going any further for the proof of our main theorem, we need two more lemmas.

**Lemma 4.4.** ([NT2]) *Let  $f \geq 0$  be a function on a complete noncompact Riemannian manifold  $M^m$  with*

$$R_{ij} \geq -(m-1)K$$

*for some  $K \geq 0$ . Let*

$$u(x, t) := \int_M H(x, y, t) f(y) dy.$$

*Assume that  $u$  is defined on  $M \times [0, T]$  for some  $T > 0$  and that for  $0 < t \leq T$ ,*

$$(4.7) \quad \lim_{r \rightarrow \infty} \exp\left(-\frac{r^2}{20t}\right) \int_{B_o(r)} f = 0.$$

*and  $p \geq 1$ ,*

$$\begin{aligned} &\frac{1}{V_o(r)} \int_{B_o(r)} u^p dx \\ &\leq C_{m,p} \left[ \frac{1}{V_o(4r)} \int_{B_o(4r)} f^p dx + \left( C_2(K, t) \int_{4r}^\infty \left( \frac{s}{\sqrt{t}} + \frac{s^2}{t} \right) \exp\left(-\frac{s^2}{40t}\right) \frac{1}{V_o(s)} \int_{B_o(s)} f d\left(\frac{s^2}{t}\right) \right)^p \right] \end{aligned}$$

*where  $C_2(K, t) = C_m t e^{C_m K t}$  and  $C_m$  is constant only depend on dimension  $M$ .*

**Lemma 4.5.** ([Li]) *Let  $(M, J, \theta)$  be a complete strictly pseudoconvex CR  $(2n+1)$ -manifold and  $f(x, t)$  be the subsolution of the heat equation satisfying*

$$\left(\frac{\partial}{\partial t} - \Delta_b\right) f(x, t) \leq 0 \text{ on } M \times [0, T]$$

*with  $f(x, 0) \leq 0$  on  $M$ . Then  $f(x, t) \leq 0$  for all  $t < T$  if there exists  $a > 0$  such that*

$$\int_0^T \int_M f^2(x, t) e^{-ar^2} d\mu(x) dt < \infty.$$

## 5. PROOF OF CR OPTIMAL GAP THEOREM

In this section, by using the CR moment type estimate (Theorem 3.1) and the linear trace LYH inequality (Theorem 4.1), we are able to prove the CR optimal gap theorem.

**Proof of the main theorem:**

**Proof.** Here is the main idea : In the following we first use proposition 3.1 to construct  $\eta_\mu$  on exhaustion domain  $\Omega_\mu$ . Schauder estimates provide the convergence of  $\eta_\mu$  (**Step 1**) to a unique solution  $\eta$ . Define  $u := tr_h \eta$  and  $u$  is a solution of sublaplacian heat equation with initial condition  $S(y)$ . By uniqueness theorem ([D]) of the nonnegative heat solution we have  $u^{(i)} \rightarrow u$ . Now this allows us in one hand using trace linear Harnack estimate on  $tr_h \eta$  to obtain monotonicity formula

$$(5.1) \quad (tu)_t \geq 0$$

which apply to every nonnegative heat solution and on the other hand using moment type estimate (which only apply to heat solution with  $P_t f$  type and  $f$  is bounded) on  $u^{(i)} := P_t \rho^{(i)} S$  to obtain that

$$u^{(i)} = o(t^{-1}).$$

Hence as well as  $u$ . Combing these results, the initial condition are forced to be zero and the gap theorem holds.

Note that we derived the monotonicity property (5.1) by lemma 4.2, 4.3, and the vanishing of mixed term in LYH quantity (4.6). The condition (1.2) is applied while we use Theorem 3.1 for  $a = -2$  to obtain

$$u = o(t^{-1}).$$

Now we split the detail proof into two steps :

(i) **Step 1 : Convergence of  $\eta_\mu^{(i)}$ :** Let  $\Omega_\mu$  be an sweetsop exhaustion domains,  $\rho^{(i)}$  be a cut-off function support in  $B(2R_i)$  such that  $0 \leq \rho^{(i)} \leq 1$ ,  $\rho^{(i)} = 1$  in  $B(R_i)$ ,  $\|\nabla_b^{m_1} \nabla_T^{m_2} \rho^{(i)}\| \leq \frac{C}{R_i}$  for  $m_1, m_2 = 0, 1, 2$ ,  $m_1 + m_2 \geq 1$  and some constant  $C$ . Note that for each  $i$ , there exists  $N_i$  such that for  $\mu \geq N_i$ ,  $B(2R_i) \subset \Omega_\mu$ . Let  $\eta_\mu^{(i)}$  be the solution as in Proposition 3.1 on  $\Omega_\mu$  for any  $\mu \geq N_i$  with initial condition  $\rho^{(i)} Ric$ . Now we define

$$u^{(i)}(x, t) := \int_M p(x, y, t) \rho^{(i)} S(y) d\mu(y).$$

Then  $u^{(i)}(x, t)$  satisfies

$$(5.2) \quad \frac{\partial}{\partial t} u^{(i)}(x, t) - \Delta_\varepsilon u^{(i)}(x, t) = -\varepsilon^2 u_{00}^{(i)}(x, t),$$

where  $\Delta_\varepsilon = \Delta_b + \varepsilon^2 T^2$  is Riemannian Laplacian with respect to the adapted metric  $h_\varepsilon := h + \varepsilon^{-2} \theta^2$ . Moreover, proposition 3.2 imply  $\eta_\mu^{(i)}(x, t)$  is nonnegative and

$$(5.3) \quad \left\| \eta_\mu^{(i)}(x, t) \right\| \leq tr_h \eta_\mu^{(i)}(x, t) \leq u^{(i)}(x, t),$$

for all  $\mu \geq N_i$ . Now we estimate  $u_{00}^{(i)}(x, t)$  first. Since  $u^{(i)}(x, t)$  is a solution of sub-Laplacian heat equation, we have

$$\frac{\partial}{\partial t} u_{00}^{(i)}(x, t) - \Delta_b u_{00}^{(i)}(x, t) = 0$$

due to vanishing torsion. We define  $l^{(i)}(x, t) = \left| u_{00}^{(i)}(x, t) \right|$ , and observe that it is a subsolution of heat equation with initial condition satisfying the followings

$$\begin{aligned} &= \left| l^{(i)}(x, t)(x, 0) \right| \\ &= \left| \nabla_T \nabla_T \rho^{(i)} S(y) \right| \\ &\leq \frac{C}{R_i} \chi_{B_{2R_i} \setminus R_i} S(y), \end{aligned}$$

where  $\chi_{B_{2R_i} \setminus R_i}(y)$  is a function with 1 in annulus  $B(2R_i) \setminus B(R_i)$  and zero elsewhere. By maximum principle  $l^{(i)}(x, t)$  is controlled by a sub-Laplacian heat solution.

Next we define

$$g(x, t) := \int_M p(x, y, t) \frac{C}{R_i} \chi_{B_{2R_i} \setminus R_i} S(y) dy.$$

By moment type estimate

$$(5.4) \quad g(x, t) = \frac{1}{R_i} o(t^{-1}),$$

where the particular coefficient in  $o(t^{-1})$  does not depend on  $i$ . To summarize, we have

$$(5.5) \quad \left| u_{00}^{(i)}(x, t) \right| = l^{(i)}(x, t) \leq g(x, t) = \frac{1}{R_i^2} o(t^{-1}).$$

We return to equation (5.2). Now we restricted on  $B(r) \times [\epsilon, T]$  and try to obtain estimate not depend on index  $i$ . Now we define

$$L^{(i)}(x, t) = u^{(i)}(x, t) + \varepsilon^2 e^{T-t} \sup_{B(r) \times [\epsilon, T]} g(x, t)$$

so that  $L^{(i)}(x, t)$  satisfy

$$(5.6) \quad \frac{\partial}{\partial t} L^{(i)}(x, t) - \Delta_\varepsilon L^{(i)}(x, t) \leq 0.$$

Applying mean value theorem (Theorem 1.2 in [LT]) to function  $L^{(i)}(x, t)$ , we have

$$\begin{aligned} \sup_{B_\varepsilon((1-\delta)r) \times [\epsilon, T]} L^{(i)} &\leq C_{16} \left\{ \frac{1}{(\delta r)^{2n+3}} \frac{V_\varepsilon\left(\frac{2}{\varepsilon^2}, 2r\right)}{V_\varepsilon(r)} \left( r \frac{\sqrt{2}}{\varepsilon} \coth\left(r \frac{\sqrt{2}}{\varepsilon}\right) + 1 \right) \exp\left(C_{17} \frac{2}{\varepsilon^2} T\right) \right\} \\ &\quad \times \int_\epsilon^T ds \int_{B_\varepsilon(r)} L^{(i)}(y, s) d\mu_\varepsilon(y) + (1 + \varepsilon_1) \sup_{B_\varepsilon(r)} L^{(i)}(\cdot, \epsilon). \end{aligned}$$

Let  $B_\varepsilon(r), d\mu_\varepsilon(y)$  denote the ball with radius  $r$  and volume element which is respected to metric  $h_\varepsilon$ . The above inequality also means

$$\begin{aligned} \sup_{B_\varepsilon((1-\delta)r) \times [\epsilon, T]} u^{(i)} &\leq C_{16} \left\{ \frac{1}{(\delta r)^{2n+3}} \frac{V_\varepsilon\left(\frac{2}{\varepsilon^2}, 2r\right)}{V_\varepsilon(r)} \left( r \frac{\sqrt{2}}{\varepsilon} \coth\left(r \frac{\sqrt{2}}{\varepsilon}\right) + 1 \right) \exp\left(C_{17} \frac{2}{\varepsilon^2} T\right) \right\} \\ (5.7) \quad &\quad \times \int_0^T ds \int_{B_\varepsilon(r)} L^{(i)}(y, s) d\mu_\varepsilon(y) \\ &\quad + (1 + \varepsilon_1) \sup_{B_\varepsilon(r)} u^{(i)}(\cdot, \epsilon) + (1 + \varepsilon_1) \varepsilon^2 e^{T-\epsilon} \sup_{B_\varepsilon(r)} g(x, \epsilon). \end{aligned}$$

We only need to estimate the first term of (5.7) below, since the other terms are bounded. We define

$$L_\varepsilon^{(i)}(y, s) := \int_M H_\varepsilon(x, y, t) \left\| \rho^{(i)} S \right\| (y) d\mu_\varepsilon(y)$$

and again we have

$$(5.8) \quad L^{(i)}(y, s) \leq L_\varepsilon^{(i)}(y, s) + \varepsilon^2 e^T \sup_{B(r) \times [\epsilon, T]} g(x, t) \leq L_\varepsilon^{(i)}(y, s) + \varepsilon^2 e^T \frac{o(t^{-1})}{R_i^2}.$$

Now the first term of (5.7) is estimated by using (5.8) and Lemma 4.4 as following

$$(5.9) \quad \begin{aligned} & \frac{1}{V_{o,\varepsilon}(r)} \int_{B_o(r)} L_\varepsilon^{(i)}(y, t) d\mu_\varepsilon(y) \\ & \leq C_{m,1} \frac{1}{V_{o,\varepsilon}(4r)} \int_{B_{o,\varepsilon}(4r)} \|\rho^{(i)} S\|(y) d\mu_\varepsilon(y) \\ & + C_m e^{C_m \frac{1}{\varepsilon^2} t} \int_{4r}^\infty \left( \frac{s}{\sqrt{t}} + \frac{s^2}{t} \right) \exp\left(-\frac{s^2}{40t}\right) \frac{1}{V_{o,\varepsilon}(s)} \int_{B_{o,\varepsilon}(s)} \|\rho^{(i)} S\|(y) d\mu_\varepsilon(y) d\left(\frac{s^2}{t}\right). \end{aligned}$$

The integral  $\int_{B_{o,\varepsilon}(4r)} \|\rho^{(i)} S\|(y) d\mu_\varepsilon(y)$  inside both terms in (5.9) are estimated by assumption (1.2) and is controlled by quantity that not depend on  $i$ . Hence (5.7) and (5.9) imply

$$(5.10) \quad \sup_{B_\varepsilon((1-\delta)r) \times [\epsilon, T]} u^{(i)} \leq C(\varepsilon, r, T, n, \rho^{(i)} S)$$

and

$$(5.11) \quad \max_{B_\varepsilon(r) \times [\epsilon, T]} tr_h \eta_\mu^{(i)}(x, t) \leq C(\varepsilon, r, T, n, \rho^{(i)} S).$$

Now the interior Schauder estimate can be applied to extract a convergent subsequence  $\eta_{\mu_k}^{(i)} \rightarrow \eta^{(i)}$  that satisfies the CR Lichnerowicz-subLaplacian heat equation on  $[0, T]$ . Note that  $tr_h \eta^{(i)}(x, 0) = u^{(i)}(x, 0)$ , and by uniqueness of bounded sub-Laplacian heat solution (from lemma 4.5) we actually have

$$tr_h \eta^{(i)}(x, t) = u^{(i)}(x, t).$$

By (5.2), (5.5), (5.10) and Schauder estimates, there is a subsequence  $u^{(i_j)} \rightarrow u$  and  $\eta^{(i_j)} \rightarrow \eta$  in any fixed compact subset with an arbitrary chosen Hölder norm (by choosing  $\beta_0$  large for sweetsop domain, see appendix). Note in (5.5) as  $i$  goes to infinity we can conclude  $\nabla_T \nabla_T u(x, t) = 0$  and similarly  $\nabla_T u(x, t) = 0$  and  $\nabla_T \eta(x, t) = 0$  by using that  $\|\eta_0^{(i)}\|$  is a subsolution of sub-Laplacian heat equation as follows

$$\left( \frac{\partial}{\partial t} - \Delta_b \right) \|\eta_0^{(i)}\| = \frac{1}{\|\eta_0^{(i)}\|} (2R_{\alpha\bar{\gamma}\mu\bar{\beta}} \eta_{0\gamma\bar{\mu}} - (R_{\gamma\bar{\beta}} \eta_{0\alpha\bar{\gamma}} + R_{\alpha\bar{\gamma}} \eta_{0\gamma\bar{\beta}}) \eta_{0\zeta\bar{\xi}} h_{\beta\bar{\zeta}} h_{\xi\bar{\alpha}}) \leq 0.$$

Here we use the facts that bisectional curvature is nonnegative and vanishing torsion. Moreover, requirement for applying maximum principle is guaranteed by similar argument as (5.3), we have

$$(5.12) \quad \|\eta_0^{(i)}\|(x, t) \leq C \int p(x, y, t) |\nabla \rho| S(y) d\mu(y) \leq \frac{C}{R_i} o(t^{-1}).$$

As  $i$  goes to infinity,  $\eta_0 = 0$ . However, by now we do not know yet through the subsequence the two functions  $tr_h \eta(x, t)$  and  $u(x, t)$  are the same even they have the same initial condition. One regards both  $u(x, t)$  and  $tr_h \eta(x, t)$  as solutions of Laplacian heat equations associated to adapted metric (due to  $\nabla_T u(x, t) = \nabla_T tr_h \eta(x, t) = 0$ ), and the manifold are seen as Riemannian manifold with Riemannian curvature bounded below by  $-\frac{1}{\varepsilon^2}$  (Theorem 4.9 in [CC1]). Now by the uniqueness of nonnegative Laplacian heat solution ([D]) on complete manifold with Riemannian Ricci curvature bounded below, we can conclude that

$$u(x, t) = tr_h \eta(x, t).$$

Note that  $u$  is the unique sub-Laplacian heat solution with  $\nabla_T u(x, t) = 0$ , and since any such  $u$  we can find a sequence of  $u^{(i)}$  that satisfy moment type estimates converge to  $u$ . Hence  $u$  satisfy the moment type estimate.

(ii) **Step 2 : Monotonicity of  $tu$  :** By our assumptions on  $Ric$ , and the upper bound of  $\eta(x, t)$  by  $u(x, t) = o(t^{-1})$ , (4.2), (4.3) and (4.4) in Theorem 4.1 are satisfied. Hence by Lemma 4.2 and (4.6),  $tr_h \eta$  satisfy

$$(5.13) \quad u_t + (\bar{\partial}_b^* \Lambda \bar{\partial}_b + conj) \phi + \frac{u}{t} \geq 0.$$

In the following we are going to prove the mixed terms  $(\bar{\partial}_b^* \Lambda \bar{\partial}_b + conj) \phi$  of (5.13) vanishing so the monotonicity

$$(5.14) \quad (tu)_t \geq 0$$

follows. Hence

$$tu(x, t) \equiv 0,$$

for any  $x$  and  $t > 0$ . The flatness then follows from  $u(x, 0) \equiv 0$ .

In fact, we first define  $\sigma^{(i)} := \Lambda \bar{\partial}_b \eta^{(i)}$  (note  $\eta^{(i)} = \frac{1}{\sqrt{-1}} \phi^{(i)}$ ). Then direct calculation shows that

$$\left( \frac{\partial}{\partial t} - \Delta_b \right) \left\| \eta_{\mu_k}^{(i)} \right\|^2 \leq -2 \left\| \nabla \eta_{\mu_k}^{(i)} \right\|^2.$$

We integrate on both sides over  $\Omega_{\mu_k}$  and apply Dirichlet condition (using boundary regularity in Proposition 3.1). After taking  $\mu_k \rightarrow \infty$ , we have

$$(5.15) \quad 2 \int_0^t \int_M \left\| \nabla_b \eta^{(i)} \right\|^2(x, s) d\mu ds \leq \int_M \left\| \eta^{(i)}(x, 0) \right\|^2 d\mu = \int_M \left\| \rho^{(i)} Ric \right\|^2 d\mu.$$

Due to  $\left\| \sigma^{(i)} \right\|(x, t) \leq \left\| \nabla_b \eta^{(i)} \right\|(x, t)$ , (5.15) and assumption (1.2) we have for some  $a' > 0$ ,

$$(5.16) \quad \int_0^t \int_M e^{-a'r^2} \left\| \sigma^{(i)} \right\|(x, s) d\mu ds < \infty.$$

By Lemma 4.3, and direct calculation shows that

$$\left( \frac{\partial}{\partial t} - \Delta_b \right) \left\| \sigma^{(i)} \right\|(x, t) \leq \left\| \sigma_0^{(i)} \right\|.$$

Since  $\left\| \sigma_0^{(i)} \right\| \leq \left\| \nabla \eta_0^{(i)} \right\|$  and  $\left\| \eta_0^{(i)} \right\|(x, t)$  satisfy 5.12, by Schauder estimates [Si] we have for any  $\tilde{\varepsilon} > 0$ , there exists  $n_{\tilde{\varepsilon}} > 0$  such that  $\left\| \sigma_0^{(i)} \right\| \leq \left\| \nabla \eta_0^{(i)} \right\| \leq \tilde{\varepsilon}$  for any  $i \geq n_{\tilde{\varepsilon}}$ . This shows that for any  $(x, t) \in [0, T)$

$$\left( \frac{\partial}{\partial t} - \Delta_b \right) \left\| \sigma^{(i)} \right\|(x, t) + e^{T-t} \tilde{\varepsilon} \leq 0.$$

We define  $v^{(i)}(x, t)$  as follow

$$v^{(i)}(x, t) = \int_M p(x, y, t) \left\| \Lambda \bar{\partial}_b (\rho^{(i)} Ric) \right\|(y) d\mu(y).$$

Duo to (5.16) and maximum principle we have

$$\left\| \sigma^{(i)} \right\|(x, t) + e^{T-t} \tilde{\varepsilon} \leq v^{(i)}(x, t) + e^{T-t} \tilde{\varepsilon}.$$

Since torsion is vanishing, it implies  $\bar{\partial}_b Ric = 0$  and by nonnegativity of Ricci curvature it follows that

$$(5.17) \quad \left\| \Lambda \bar{\partial}_b (\rho^{(i)} Ric) \right\|(y) \leq \frac{C}{R_i} \chi_{B_{2R_i} \setminus R_i} S(y).$$



Similarly as (5.4), (5.17) gives that  $v^{(i)} \rightarrow 0$  uniformly on any compact subset as  $i \rightarrow \infty$ . Since  $\tilde{\varepsilon}$  is arbitrary, we have

$$\|\sigma\|(x, t) = 0.$$

Finally, as a result we have  $(tu)_t \geq 0$  and then  $u = o(t^{-1})$ . This completes the proof.  $\square$

## APPENDIX A.

In this appendix, we construct "nice" domains to avoid the possibility of the bad regularity for heat solutions in the case of degenerated parabolic systems. In fact, we will give a proof on existence and regularity result for  $(1, 1)$ -form  $\phi$  of the Lichnerowicz-subLaplacian heat equation. In the proof of main theorem, one required some regularity of the heat solution in order to prove the mixed terms  $(\bar{\partial}_b^* \Lambda \bar{\partial}_b + \text{conj}) \phi$  of (5.13) vanishing (then the monotonicity follows). While we construct heat solution on complete manifolds with exhaustion domains, we need the interior regularity at least  $C^{2,\alpha}(\Omega_\mu)$  and boundary regularity as continuous function in  $C(\bar{\Omega}_\mu)$ . This requirement are needed for Arzela Ascoli theorem and integration by part in (5.15). In semigroup method, better regularity of evolution equation comes from the regularity of infinitesimal generator.

We denote  $C^{2,\alpha}(\Omega, \Lambda^{1,1})$  as  $C^{2,\alpha}$  sections of  $\Lambda^{1,1}$  on bounded domain  $\Omega$ . In our case, it is  $\Delta_H$  on Banach space  $C^{2,\alpha}(\Omega, \Lambda^{1,1}) \cap C(\bar{\Omega}, \Lambda^{1,1})$ . Note  $\Delta_H = -\frac{1}{2}(\square_b + \bar{\square}_b)$ . Here we denote  $u$  as solution of following Dirichlet problem

$$(A.1) \quad \square_b \phi = g$$

for  $g \in C^\infty(\Omega, \Lambda^{1,1})$ . First we state some results :

**Theorem A.1** (Kohn). *Let  $M$  be a strictly pseudoconvex CR  $(2n+1)$ -manifold. If  $1 \leq q \leq n-1$ , then  $\|\phi\|_{\frac{1}{2}}^2 \leq C[(\square_b \phi, \phi) + \|\phi\|_0^2]$  for  $\phi \in C^\infty(\Lambda^{0,q})$ .  $\|\cdot\|_s$  stands for the  $L^2$  Sobolev norm of order  $s$ .*

**Remark A.1.** 1. *From the hypothesis in above theorem it requires  $n \geq 2$ . When  $n = 1$ , one refers to [J1].*

2. *Even though the operator  $\square_b$  is not  $\Delta_H$ , in [J2] (see p.146) they actually prove the case for  $\alpha = 0$ . Moreover, we have  $\Delta_H = \mathcal{L}_\alpha$  with  $\alpha = 0$  up to lower order terms. Here  $\mathcal{L}_\alpha = -\Delta_b + i\alpha T$ .*

The following is the interior and boundary regularity result by Jerison [J1].

**Theorem A.2.** *Let  $U$  be the open subset of  $M$  containing no characteristic points of  $\partial\Omega$ . If  $\psi, \varphi \in C_0^\infty(U)$ ,  $\psi = 1$  in the neighborhood of the support of  $\varphi$ , and  $u$  satisfies (A.1) with  $\psi g \in \Gamma_\beta(\bar{\Omega}, \Lambda^{0,q})$ , then  $\varphi \phi \in \Gamma_{\beta+2}(\bar{\Omega}, \Lambda^{0,q})$  and*

$$\|\varphi \phi\|_{\Gamma_{\beta+2}} \leq c \left( \|\psi g\|_{\Gamma_\beta} + \|\psi \phi\|_{L^2} \right).$$

When an isolated characteristic boundary point occurs, Jerison proved the regularity result when the neighborhood have strictly convexity property. The convexity is defined by Folland-Stein local coordinates  $\Theta(p, -) : U \rightarrow \mathbb{H}^n$ , and the boundary near point  $p$  is corresponding to graph  $\tilde{t} = \sum \alpha_i \tilde{x}_i^2 + \beta_j \tilde{y}_j^2 + e(\tilde{x}, \tilde{y})$ , where  $e(\tilde{x}, \tilde{y}) = O(|\tilde{x}|^3 + |\tilde{y}|^3)$ . Strictly convex means  $\alpha_i, \beta_j > 0$  (see eq. (7.4) and A.3 in [J2]). In the following we state the theorem in the form

we want. Reader who is confused can refer to theorem 7.6, Proposition 7.11, and Corollary 10.2 in [J2].

**Theorem A.3.** *Let  $p$  be an isolated characteristic point on  $\partial\Omega$  and in some neighborhood  $U_p$  of  $p$  the geometry  $U_p \cap \Omega$  is like the domain  $\{(x, y, t) : M_c(|x|^2 + |y|^2) < t\}$  in the Heisenberg group, where  $M_c$  a positive number. Then  $\varphi\phi \in \Gamma_{\beta+2}(\bar{\Omega}, \Lambda^{0,q})$ , where the best  $\beta$  depends on  $M_c$ . Moreover, as  $M_c \nearrow \infty$ , one can choose  $\beta \nearrow \infty$ .*

**Remark A.2.** *In Theorem A.3, one required  $g \in \Gamma_\beta(\bar{\Omega}, \Lambda^{0,q})$  for  $\beta > 2$ . Moreover,  $\beta$  has upper bound  $\beta_0 - 2$ , where  $\beta_0$  is an index related to the geometry of the boundary. In [J2], they proved  $M_c \nearrow \infty$ , then  $\beta_0 \nearrow \infty$ .*

In order to construct a  $C^{2,\alpha}$  Lichnerowicz-subLaplacian heat solution, we need the exhaustion domain which satisfy the property above. In the following we prove that it is possible by perturbing the boundary of exhaustion domain.

**Theorem A.4.** *For any given positive number  $M_c$ , there exists exhaustion domains  $\Omega_\mu$  such that  $\partial\Omega_\mu$  consist only isolated characteristic points with property as in Theorem A.3 with given  $M_c$ .*

**Proof.** We construct the exhaustion domain with smooth boundary arbitrarily. Since  $\partial\Omega_\mu$  is compact, we define  $\Xi_\mu$  the set consisting all the characteristic points. Then the closure of  $\Xi_\mu$  is compact. At each point there exist coordinate  $V_p$  such that we can express the boundary as  $r(z, t) = t - q(z) + e(x, y)$  in  $B_p(\varepsilon_p)$  for some  $\varepsilon_p$  depend on  $p$ , where  $q(z) = \alpha_i x_i^2 + \beta_j y_j^2$  for some real numbers  $\alpha_i, \beta_j$ . Since injective radius (with respect to some adapted metric) is uniformly bounded below on  $\partial\Omega_\mu$ ,  $\varepsilon_p$  can be chosen to not depend on  $p$  but  $\mu$  only. These Folland-Stein coordinate neighborhoods form an open covering for  $\bar{\Xi}_\mu$ .

Now we claim there is a small modification to boundary so that  $\bar{\Xi}_\mu$  contains only isolated characteristic points.

Assume  $B_{p_i}(\varepsilon)$  are the covering of  $\bar{\Xi}_\mu$ , we can choose  $\varepsilon_1 < \varepsilon_2 < \varepsilon$  such that  $B_{p_i}(\varepsilon_1)$  are still a covering of  $\bar{\Xi}_\mu$ . We start at point  $p_1$ . First we deform the graph in the coordinate of  $B_{p_1}(\varepsilon_1)$  to plane  $t = 0$  and smoothly attached to graph on  $\partial B_{p_1}(\varepsilon_2)$ . Under the deformation we keep point  $p_1$  as the only characteristic point. This is possible by noticing that we only need to take  $q(z)$  into consideration (because this term dominate all the other inside small ball.) and we only need to consider the case in the Heisenberg group with graph  $t = q(z)$  in  $B_{p_i}(\varepsilon)$ . We modify  $q(z)$  into new one  $\tilde{q}(z)$  by define  $\tilde{q}(z) = -\max_{|z|=\varepsilon_2} q(z)$  in  $B_{p_i}(\varepsilon_1)$  and

$\varphi(|z|, \theta)$  in  $B_{p_i}(\varepsilon_2) \setminus B_{p_i}(\varepsilon_1)$  where  $\varphi(|z|, \theta)$  is a smooth monotone function in  $|z|$  for each  $\theta$  such that the function smoothly attached to the value  $q(z)$  on  $\partial B_{p_i}(\varepsilon_2)$  and  $\tilde{q}(z) = q(z)$  on  $B_{p_i}(\varepsilon) \cap B_{p_i}^c(\varepsilon_2)$ . This modification clearly imply the origin is the only characteristic point in  $B_{p_i}(\varepsilon_1)$ . Moreover, we can choose  $\varphi(|z|, \theta)$  very steep so that all the point  $(z, q(z))$  for  $z \in B_{p_1}(\varepsilon_2) \setminus (0, 0)$  are noncharacteristic. We define the new domain as  $\Omega_{\mu,1}$ . Specifically,

$$\Omega_{\mu,1} = \{\Omega_\mu \setminus B_{p_1}(\varepsilon_2)\} \cup (M \cap \{(z, t) : t > \tilde{q}(z) - R(z, t) \text{ for } z \in B_{p_1}(\varepsilon_2)\}).$$

Then we continue the same process on  $p_2$ , and the new domain is  $\Omega_{\mu,2}$ . Observe that the process do not create new characteristic points but eliminate all the characteristic point

inside  $B_{p_i}(\varepsilon_2)$  except  $p_i$ . Continuing this process we are able to deform domain  $\Omega_\mu$  into new one that only consist isolated characteristic points on the boundary with  $M_c = 0$ .

To modify  $M_c$  into any value we want is easier. One can do the same process by deforming the graph into parabolic.  $\square$

For convenience, we call the domain in above theorem as sweetsop domain.

**Remark A.3.** *The above theorem can be simplified if we can construct strictly convex domain in  $M$ . But the existence to this kind of exhaustion domain isn't known yet.*

We recall theorems from semigroup method. For the definition of analytic semigroup, one can refer to definition 12.30 in [R] ([P]). We cited the characterization of infinitesimal generator of analytic semigroups. Notation here  $X$  is Banach space and  $A$  is operator defined on  $X$ . Note  $A$  can be unbounded operator  $A : D(A) \rightarrow X$ , where  $D(A)$  is a subset in  $X$  such that  $Ax$  can be defined. As before, we denote  $\Gamma_\beta$  the Lipschitz classes associated to nonisotropic distance (referred [J2]) and  $\Gamma_\beta(\bar{\Omega}, \Lambda^{1,1})$  the restriction to  $\bar{\Omega}$  of sections of  $\Lambda^{1,1}$  with coefficients in  $\Gamma_\beta(\bar{\Omega})$ . We denote  $\|\cdot\|_{\Gamma_\beta}$  the norm of Banach space  $\Gamma_\beta(\bar{\Omega}, \Lambda^{1,1})$ , and  $R_\lambda(A)$  as the inverse operator of  $A_\lambda := A - \lambda I$  as  $A_\lambda$  is one-to-one. The resolvent set of the operator  $A$  is the subset of  $\mathbb{C}$  that  $R_\lambda(A)$  exists, bounded, and the domain is dense in  $X$ . When we apply, we let  $X = \Gamma_\beta(\bar{\Omega}, \Lambda^{1,1})$  and  $A = \Delta_H$ . Here we state general theorems for following evolution systems

$$\dot{u} = Au + f$$

where  $f \in X$ .

**Theorem A.5.** ([R, Theorem 12.31]) *A closed, densely defined operator  $A$  in  $X$  is the generator of an analytic semigroup if and only if there exists  $\omega$  a real number such that the half-plane  $\operatorname{Re} \lambda > \omega$  is contained in the resolvent set of  $A$  and, moreover, there is a constant  $C$  such that*

$$(A.2) \quad \|R_\lambda(A)\| \leq \frac{C}{|\lambda - \omega|}$$

for  $\operatorname{Re} \lambda > \omega$  and  $\|\cdot\|$  is the norm of  $X$ .

**Theorem A.6.** ([R, Theorem 12.33]) *Let  $A$  be the infinitesimal generator of an analytic semigroup and assume that the spectrum of  $A$  is entirely to the left of the line  $\operatorname{Re} \lambda = \omega$ . Then there exists a constant  $M$  such that*

$$\|e^{At}\| \leq Me^{\omega t},$$

where  $\|\cdot\|$  is the norm of  $X$ .

One can refer to section 7.1 in [P] or page 421 in [E] for the application of semigroup theory for  $A$  is a strong elliptic operator. In our case, the missing boundary regularity is replaced by theorem A.3 (refer to [GV], [GV2]). Then one can follow Stewart [SH] and consider Hölder spaces as interpolation space [LA] to obtain the resolvent estimates A.2. As a result, the regularity of the parabolic systems follows by theorem A.6.

In conclusion, we are able to choose exhaustion domain with  $\beta_0$  large enough, then follow theorem above, we can choose  $\beta$  large enough to make sure the function space  $X$  is contained

in  $C^{2,\alpha}$ . This is possible by relation  $C^\beta \subset \Gamma_\beta \subset C^{\beta/2}$  as in 20.5, 20.6 of [FS]. This completes the proof of Proposition 3.1.

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